Written test of Advanced Quantum Mechanics

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Exam time: 2 hours. You can use the Clebsch-Gordan sheet by PDG.

ESERCISE 1

Consider a spin-$\frac{1}{2}$ particle constrained to move on a sphere of radius R and subject to the following Hamiltonian:

$$H = \frac{\omega}{\hbar}\left(\mathbf{J}^2 + \mathbf{L}^2 + \frac{3}{2}\hbar J_z\right), \qquad \text{where } \omega > 0 \text{ and } \mathbf{J} = \mathbf{L} + \mathbf{S}.$$

1. Determine the eigenvalues and the eigenkets of the Hamiltonian for energies $E < 4\hbar\omega$, and discuss the degeneracy.

2. Write the states $|\psi\rangle$ that satisfy the condition $H|\psi\rangle = \frac{7}{2}\hbar\omega|\psi\rangle$. Among these states, identify those for which the probability of measuring $J_z = \hbar/2$ is equal to the probability of measuring $J_z = -3\hbar/2$.

3. For the states identified in point 2, a measurement of $\mathbf{L}^2$ and L_z is performed. Which values can be found, and with what probabilities?

EXERCISE 2

Two identical spin-$\frac{1}{2}$ particles in the center-of-mass reference frame have as Hamiltonian

$$H = \frac{\mathbf{p}_1^2}{2m} + \frac{\mathbf{p}_2^2}{2m} + \frac{1}{4}m\omega^2(\mathbf{r}_1 - \mathbf{r}_2)^2 + \gamma L_z,$$

with $0 < \gamma \ll \omega$. Here $\mathbf{L}$ is the total orbital angular momentum of the system.

1. Determine the eigenvalues E (together with their degeneracies) of the Hamiltonian that satisfy $E < 4\hbar\omega$.

2. The system is in a state $|\psi\rangle$ such that an energy measurement E can only yield $E \leq 3\hbar\omega$; moreover $\mathbf{L}^2|\psi\rangle = 2\hbar^2|\psi\rangle$ and $\mathbf{J}^2|\psi\rangle = 0$, where $\mathbf{J} = \mathbf{L} + \mathbf{S}_1 + \mathbf{S}_2$. Determine $|\psi\rangle$.

3. In the state $|\psi\rangle$, what possible values can be obtained by a measurement of the z component of the spin of one particle? With what probabilities?