Written test of Advanced Quantum Mechanics

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Exam time: 2 hours. You can use the Clebsch-Gordan sheet by PDG.

ESERCISE 1

Consider two identical particles A and B with spin 1/2, constrained to move on a sphere of radius R. The Hamiltonian of the system is

$$H = 2\alpha J^2 + \alpha L_A^2 + \alpha L_B^2,$$

with $\alpha > 0$, $\mathbf{J} = \mathbf{L}_{\text{tot}} + \mathbf{S}_{\text{tot}}$, $\mathbf{L}_{\text{tot}} = \mathbf{L}_A + \mathbf{L}_B$, and $\mathbf{S}_{\text{tot}} = \mathbf{S}_A + \mathbf{S}_B$.

1. Determine the ground state ψ_0 and the first excited state ψ_1 of the system. Compute its energy and express them as a linear combination of

$$|\ell_A m_A\rangle |\ell_B m_B\rangle |s, s_z\rangle,$$

   where $|\ell_A m_A\rangle$ are the eigenfunctions of L_A^2 and L_{Az}, $|\ell_B m_B\rangle$ are the analogous eigenfunctions for particle B, and $|s, s_z\rangle$ are the spinors for the total spin $\mathbf{S}_{\text{tot}}$.

2. A measurement of the z component of the spin of one of the two particles is performed on the states ψ_0 and ψ_1. What are the possible outcomes of the measurement? With what probabilities?

ESERCISE 2

A particle of mass m, charge e and spin 1/2 is subject to the following Hamiltonian:

$$\mathcal{H} = \frac{\mathbf{p}^2}{2m} - \frac{e^2}{r} + \alpha \frac{\mathbf{J}^2}{\hbar^2},$$

with $\alpha > 0$, $\mathbf{J} = \mathbf{L} + \mathbf{S}$ is the total angular momentum of the system.

1. Consider the following spinor wavefunctions:

$$\Psi_a = \frac{N_a}{r} R_{21}(r) \begin{pmatrix} x + iy \\ 0 \end{pmatrix}, \qquad \Psi_b = \frac{N_b}{r} R_{21}(r) \begin{pmatrix} 0 \\ z \end{pmatrix},$$

where N_i are normalization constants, the symbols n, l and $R_{n,l}(r)$ refer to the usual hydrogen atom theory, and the spinors refer to the z axis. Rewrite them as linear combinations of tensor products $|n, l, l_z\rangle \otimes |s, s_z\rangle$.

2. State which of these wavefunctions are energy eigenstates and with which eigenvalue.

3. Compute the normalizations N_a, N_b and determine the wavefunctions $\Psi_i(t)$ obtained by time evolution from Ψ_i for $t > 0$ (it is convenient to use a basis in which $\mathbf{J}^2$ and J_z are diagonal).

Hint: The eigenstates of the hydrogen atom are $R_{nl}(r)Y^l_{l_z}(\theta, \phi)$ and have energy $E_n = -\frac{me^4}{2\hbar^2 n^2}$.
Hint 2: Remember $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$.