Written test of Advanced Quantum Mechanics

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Exam time: 2 hours. You can use the Clebsch-Gordan sheet by PDG.

ESERCISE 1

The state of an electron in a hydrogen atom with orbital angular momentum $l = 1$ and spin 1/2 is described by the normalized wave function

$$\psi(r, \phi, \theta) = \frac{R_{2,1}(r)}{\sqrt{2}} \left[Y_{1,0}(\phi, \theta) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + Y_{1,1}(\phi, \theta) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right],$$

where $R_{2,1}(r)$ is the radial wave function for the hydrogen atom with $n = 2$ and $l = 1$, and

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

correspond to the spin states $|1/2, +1/2\rangle$ and $|1/2, -1/2\rangle$, respectively.

1. Determine the possible outcomes of a measurement of J^2 and J_z, where $\mathbf{J} = \mathbf{L} + \mathbf{S}$, and the corresponding probabilities.

2. Compute the matrix element $\langle \psi | \mathbf{L} \cdot \mathbf{S} | \psi \rangle$.

3. Determine the possible outcomes of a measurement of L_x and the corresponding probabilities. Explain why, after a measurement of L_x, the state remains an eigenstate of L^2 and specify the corresponding eigenvalue.

ESERCISE 2

A system is composed of two identical spin-1 particles constrained to move in three dimension. The system is described by the Hamiltonian

$$H = \frac{\boldsymbol{p}_1^2 + \boldsymbol{p}_2^2}{2m} + \frac{m\omega^2}{2} \left(\boldsymbol{r}_1^2 + \boldsymbol{r}_2^2 \right).$$

1. Determine the eigenvalues corresponding to the first three energy levels and discuss their degeneracy.

2. Determine a basis of eigenvectors corresponding to the three levels found above, such that they are simultaneous eigenstates of the Hamiltonian and of the operators S^2 and S_z, with $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$.