Written test of Advanced Quantum Mechanics

Alessandro Pilloni

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Exam time: 2 hours. You can use the Clebsch-Gordan sheet by PDG.

EXERCISE 1

A particle of spin 1/2 and mass m is constrained to move on the surface of a sphere of radius R. At time $t=0$ the state of the particle is described by the spinor

$$|\psi(t=0)\rangle \;=\; A\begin{pmatrix}\sin\theta\,\sin\varphi\\ \sqrt{2}\,\cos\theta\end{pmatrix},$$

where A is a normalization constant, and the upper and lower components refer to the eigenstates of S_z with eigenvalues $+\hbar/2$ and $-\hbar/2$, respectively. The Hamiltonian of the system is

$$H \;=\; \frac{\vec{L}^{\,2}}{2mR^2} \;+\; \frac{\omega}{\hbar}\,\vec{L}\cdot\vec{S}\,.$$

1. Determine the normalization constant A. Determine, at time $t=0$, the possible outcomes of a measurement of $\vec{L}^{\,2}$, L_z and S_z, and the corresponding probabilities;

2. Determine, at time $t=0$, the possible outcomes of a measurement of $\vec{J}^{\,2}$ and J_z, where as usual $\vec{J}=\vec{L}+\vec{S}$, and the corresponding probabilities;

3. Determine the value of L_z at the generic time t.

EXERCISE 2

Two identical particles of spin 1/2 and mass m are constrained to move on the surface of a sphere of radius R. Their Hamiltonian is

$$H = \frac{\omega}{\hbar}\left[\vec{L}_1^{\,2}+\vec{L}_2^{\,2}\right] + \frac{\varepsilon}{\hbar^2}\,\vec{S}_1\cdot\vec{S}_2\,, \qquad 0<\varepsilon\ll\hbar\omega\,,$$

where $\vec{L}_{1,2}$ and $\vec{S}_{1,2}$ are, respectively, the orbital angular momenta and the spins of particles 1 and 2. Note that the system is not in the center of mass frame.

1. Determine the eigenvalues of H with energy $E < 3\hbar\omega$ and their degeneracies, taking into account the statistics of the particles;

2. At time $t = 0$, a measurement of $\vec{L}_1^2$ and $\vec{L}_2^2$ gives with certainty $2\hbar^2$ for both particles; a measurement of L_{1z} and L_{2z} gives with certainty $+\hbar$ for one particle and $-\hbar$ for the other; a measurement of the total spin $\vec{S}^2$, with $\vec{S} = \vec{S}_1 + \vec{S}_2$, gives with certainty 0. Show that these conditions determine the state uniquely, and write it explicitly;

3. For the state of point 2., determine the possible outcomes of a measurement of $\vec{L}_T^2$ and J_{Tz} (just the z component!), where $\vec{L}_T = \vec{L}_1 + \vec{L}_2$ and $\vec{J}_T = \vec{L}_T + \vec{S}$, and the corresponding probabilities. Determine the state at the generic time t, and say whether the above probabilities depend on time.