Gauge theories in anti-selfdual variables

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Outline

Introduction

It is conceivable that a truly deep advance in theoretical physics would involve writing down QED without writing A_m

A. Zee

We want explore possible change of variables in gauge theories

driven by the fact that some (non-perturbative) properties of gauge theories look more natural with a clever choice of the variables

There is no general theory of changes of variables in functional integrals, only simplest cases have been formally studied (Anselmi, EPJ **C73**, 2338)

ASD curvature

In particular, we are interested in the (non-linear) map from the gauge-fixed connection to the anti-selfdual curvature

$$A_m \to F^-_{mn}(A)$$

with

$$F^-_{mn} = F_{mn} - {}^*F_{mn} = F_{mn} - \tfrac{1}{2}\varepsilon_{mnrs}F_{rs}$$
$$F_{mn} = \partial_m A_n - \partial_n A_m + i\,[A_m, A_n]$$

The map is (locally) 1-to-1:

- 4 components of A_m − 1 gauge fixing condition
- 6 component of skew-symmetric F^-_{mn} − 3 *ASD* conditions

ASD curvature

Yet, an *ASD* tensor lives in the $(1,0)$ representation of the Euclidean rotation group: F^-_{mn} contains a (chiral) 3-vector

$$F^-_{mn} = \begin{pmatrix} 0 & -E_3 + H_3 & E_2 - H_2 & E_1 - H_1 \\ E_3 - H_3 & 0 & -E_1 + H_1 & E_2 - H_2 \\ -E_2 + H_2 & E_1 - H_1 & 0 & E_3 - H_3 \\ -E_1 + H_1 & -E_2 + H_2 & -E_3 + H_3 & 0 \end{pmatrix}$$

F^-_{mn} is gauge-covariant ($F^g = g^{-1} F g$)

- eigenvalues are gauge-invariant
- nice property for non-perturbative applications (compute correlators like $\langle \mathrm{tr}\, F^{-2}_{mn}(x)\, \mathrm{tr}\, F^{-2}_{mn}(0) \rangle$...)

Purpose of the talk

We want to establish that gauge theories
usually formulated in terms of the connection A_m
are perturbatively equivalent to gauge theories
formulated in terms of the *ASD* curvature F^-_{mn}

We evaluate the 1-loop effective action of the mapped theory
and show that is identical to the original one

In particular, the 1-loop β function
of the mapped theory coincides with the original one

Nicolai map

A map to *ASD* variables was explored many years ago
in *SUSY* context.

Nicolai proved that in any supersymmetric theory there exists a change of variables that sets the lagrangian in gaussian form

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DFFV explored $\mathcal{N} = 1$ *SUSY YM*: such a map is indeed the change of variables to the *ASD* curvature

$$\mu_{mn} = F^-_{mn}(A) = \left(\delta_{mr}\delta_{ns} - \tfrac{1}{2}\varepsilon_{mnrs}\right)\left(\partial_{[m,}A_{n]} + i\,[A_m, A_n]\right)$$

in the light-cone gauge $A^+ = A_0 + A_3 = 0$

This definition of F^- and of the light-cone are consistent in $(2,2)$ signature because F^- becomes complex in Minkowskian signature

$\mathcal{N} = 1$ SUSY YM

The partition function is

$$Z = \int \delta A\, \delta\lambda\, \delta\bar{\lambda}\, \exp\left[-\frac{1}{4Ng_W^2} \int d^4x \, \mathrm{tr}\left(F_{mn}^2 + \bar{\lambda} \not{D} \lambda \right) \right] \Bigg|_{A^+=0}$$

We use $\mathrm{tr}\, F_{mn}^2 = \frac{1}{2}\, \mathrm{tr}\, F_{mn}^{-2} + \mathrm{tr}\, F^*F$, and $\int d^4x \, \mathrm{tr}\, F^*F = 2NQ\,(4\pi)^2$

$$Z = \int \delta A\, \delta\lambda\, \delta\bar{\lambda}\, e^{-\frac{(4\pi)^2 Q}{2g_W^2}} \exp\left[-\frac{1}{4Ng_W^2} \int d^4x \, \mathrm{tr}\left(\tfrac{1}{2} F_{mn}^{-2} + \bar{\lambda} \not{D} \lambda \right) \right] \Bigg|_{A^+=0}$$

Integrating over $\lambda, \bar{\lambda}$ we get

$$Z = \int \delta A\, e^{-\frac{(4\pi)^2 Q}{2g_W^2}} \exp\left[-\frac{1}{8Ng_W^2} \int d^4x \, \mathrm{tr}\, F_{mn}^{-2} \right] \mathrm{Det}\, \not{D} \Bigg|_{A^+=0}$$

$\mathcal{N} = 1$ *SUSY YM*

We change variables, a Jacobian occurs

$$Z = \int \delta\mu \, e^{-\frac{(4\pi)^2 Q}{2g_W^2}} \exp\left[-\frac{1}{8Ng_W^2}\int d^4x \ \mathrm{tr}\,\mu_{mn}^2\right] \mathrm{Det}\, \not{D} \ \mathrm{Det}\,\frac{\delta A}{\delta\mu}\bigg|_{A^+=0}$$

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The Jacobian in light-cone gauge reads (DFFV)

$$\mathrm{Det}\,\frac{\delta A}{\delta\mu}\Big|_{A^+=0} = \left(\mathrm{Det}\,\not{D}\right)^{-1}$$

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- No interaction appears in partition function – good!
- For $Q \neq 0$ no renormalization occurs, $\beta(g) \equiv 0$ – bad!

Some (seemingly desperate) questions

- A faithful map must conserve the same properties as the original theory, but the β function changes!
- Where is the weak point of the argument? Light-cone gauge?
- Can we trust non-linear changes of variables?
- No cancellation occurs outside light-cone gauge and $\mathcal{N} = 1$ *SUSY YM*, can we explore further?

The Nicolai map was labelled as a formal relation, and forgotten

Non-perturbative β function of $\mathcal{N} = 1$ *SUSY YM*

The puzzle of the β function of Nicolai map for $Q \neq 0$
was solved in 2010 by Bochicchio (talk at GGI, arXiv:1107.4320)

In the latter partition function, we have supposed
the map is 1-to-1 everywhere

Non-perturbative β function of $\mathcal{N}=1$ *SUSY YM*

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If not, we must take into account separately the zero modes of the Jacobian of the map

$$Z = \int \delta\mu \, e^{-\frac{(4\pi)^2 Q}{2g_W^2}} \exp\left(-\frac{1}{8Ng_W^2}\int d^4x \, \mathrm{tr}\left(\mu_{mn}\right)^2\right)$$

$$\Lambda^{n_b[A]-\frac{1}{2}n_f[A]} \int_{\mathcal{M}} \frac{\mathrm{Pf}\left\langle \frac{\delta A(\mu)}{\delta m}, \frac{\delta A(\mu)}{\delta m}\right\rangle}{\mathrm{Pf}\left\langle \eta\left(\mu\right), \eta\left(\mu\right)\right\rangle}\Bigg|_{A^+=0}$$

Non-perturbative β function of $\mathcal{N} = 1$ *SUSY YM*

Z vanishes unless we insert the correct number of fermionic zero modes

$$Z = \int \delta A\, \delta\lambda\, \delta\bar{\lambda}\, \exp\left[-\frac{1}{4Ng_W^2}\int d^4x\, \mathrm{tr}\left(F_{mn}^2 + \bar{\lambda}\not{D}\lambda\right)\right] \lambda\cdots\lambda$$

The only observable we can evaluate with this non-perturbative approach is the gluino condensate

Anyway, it is enough to extract the β function

We will reproduce *NSVZ* β function and we will demonstrate the gluino condensate is localized on instantons

The localization on instantons

We make the functional integral computable
by cohomological localization

In the language of differential forms

$$d\omega = 0 \qquad\qquad d^2 = 0$$

then

$$Z = \int \exp[\omega] = \int \exp[\omega + d\alpha]$$

We can add an exact form to a closed form without changing the cohomology class and the value of the integral
We can simplify the action by dropping all exact forms!

In $\mathcal{N} = 1$ *SUSY YM*, the existence of Nicolai map implies the existence of a nilpotent charge Q

The localization on instantons

In the original Z in terms of A_m, we can introduce anticommuting auxiliary fields ρ_{mn} and η_r and rewrite the gluino determinant

$$\int d\lambda\, d\bar{\lambda}\, \exp\left[-\frac{1}{8Ng_W^2}\int d^4x\, \bar{\lambda}\not{D}\lambda\right] = \mathrm{Det}\, \not{D}$$

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$$\mathrm{Det}\, \not{D} = \underbrace{\mathrm{Det}\, \frac{\delta F^-}{\delta A}}_{\text{Inverse of Nicolai Jacobian}} = \int d\rho\, d\eta\, \exp\left[\frac{i}{8Ng_W^2}\int d^4x\, \rho_{mn}\frac{\delta F^-_{mn}}{\delta A_r}\eta_r\right]$$

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$$\text{Det}\,\not{D} = \text{Det}\,\frac{\delta F^-}{\delta A} = \int d\rho\, d\eta\, \exp\left[\frac{i}{8Ng_W^2}\int d^4x\, \rho_{mn}\frac{\delta F^-_{mn}}{\delta A_r}\eta_r\right]$$

Finally we introduce a commuting auxiliary field E

$$Z_\eta = \int \delta E\, \delta A\, \delta\rho\, \delta\eta\, e^{-\frac{(4\pi)^2 Q}{2g_W^2}}\, \eta \ldots \eta$$
$$\exp\left[-\frac{1}{8Ng_W^2}\int d^4x\, \text{tr}\left(E^2_{mn} + iE_{mn}F^-_{mn} - i\rho_{mn}\frac{\delta F^-_{mn}}{\delta A_r}\eta_r\right)\right]$$

The localization on instantons

In this form, Z enjoys the Parisi-Sourlas *BRS* symmetry

$$Q_{\text{BRS}}\, A = \eta \qquad Q_{\text{BRS}}\, \eta = 0 \qquad Q_{\text{BRS}}\, \rho = E \qquad Q_{\text{BRS}}\, E = 0$$

$$Q_{\text{BRS}}^2 = 0 \qquad E^2 = Q_{\text{BRS}}\,(\rho E)$$

and we can drop the E^2 term

$$Z_\eta = \int \delta E\, \delta A\, \delta\rho\, \delta\eta\, e^{-\frac{(4\pi)^2 Q}{2g_W^2}}\, \eta \dots \eta$$
$$\exp\left[-\frac{1}{8Ng_W^2}\int d^4x\, \mathrm{tr}\left(iE_{mn}F^-_{mn} - i\rho_{mn}\frac{\delta F^-_{mn}}{\delta A_r}\eta_r\right)\right]$$

The localization on instantons

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$$Q_{\mathrm{BRS}}\, A = \eta \qquad Q_{\mathrm{BRS}}\, \eta = 0 \qquad Q_{\mathrm{BRS}}\, \rho = E \qquad Q_{\mathrm{BRS}}\, E = 0$$

$$Q^2_{\mathrm{BRS}} = 0 \qquad E^2 = Q_{\mathrm{BRS}}\,(\rho E)$$

and we can drop the E^2 term

$$Z_\eta = \int \delta A \exp\left(-\frac{(4\pi)^2\, Q}{2 g_W^2}\right) \delta\left(F^-_{mn}\right) \mathrm{Det}\left(\frac{\delta F^-_{mn}}{\delta A_r}\right) \dots$$

Z is localized on instantons

Wilsonian β function of $\mathcal{N} = 1$ *SUSY YM*

Going back to the mapped Z, we know the number of zero modes around instantons backgrounds!

$$Z_\eta = e^{-\frac{(4\pi)^2 Q}{2g_W^2}} \Lambda^{n_b[A]-\frac{1}{2}n_f[A]} \int_{\mathcal{M}} \frac{\mathrm{Pf}\left\langle \frac{\delta A(\mu)}{\delta m}, \frac{\delta A(\mu)}{\delta m} \right\rangle}{\mathrm{Pf}\left\langle \eta\left(\mu\right), \eta\left(\mu\right)\right\rangle}$$

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We exponentiate the cutoff and get the expression for the renormalized constant

$$\frac{(4\pi)^2 Q}{2g_W^2(\Lambda)} - (4\pi)^2 Q \frac{3N \log \Lambda}{(4\pi)^2} = \frac{(4\pi)^2 Q}{2g_W^2(\mu)}$$

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We exponentiate the cutoff and get the expression for the renormalized constant

$$\beta(g_W) = -\frac{3N}{(4\pi)^2} g_W^3$$

Canonical β function of $\mathcal{N}=1$ *SUSY YM*

From the one-loop exact $\beta(g_W)$ we derive the *NSVZ* formula by rescaling the fields in canonical form

$$Z = \exp\left(-\frac{(4\pi)^2\, Q\,(gA_c)}{2g_W^2}\right)\Lambda^{n_b-\frac{n_f}{2}}\, g^{n_b-n_f}\int_{\mathcal{M}_Q}\frac{\mathrm{Pf}\left\langle\frac{\delta A_c}{\delta m},\frac{\delta A_c}{\delta m}\right\rangle}{\mathrm{Pf}\left\langle\eta_c,\eta_c\right\rangle}$$

We include extra-powers of g in a non-analytic redefinition of g

$$\frac{1}{2g_W^2}=\frac{1}{2g^2}+\frac{2N}{(4\pi)^2}\log g$$

and get

$$\beta(g)=\frac{-\frac{3N}{(4\pi)^2}g^3}{1-\frac{2N}{(4\pi)^2}g^2}$$

Summary on $\mathcal{N} = 1$ *SUSY YM*

- In ASD variables, the localization on instantons emerges naturally with a real non-perturbative argument
- In usual variables, the cancellation of non-zero-modes contibutions was explicitly found at 1-loop (D'Adda,Di Vecchia, PRL73B(1978)) and then argued at any order (NSVZ, NPB 229 (1983))
- This change of variables has given some deeper understanding of the theory

Perturbative equivalence (this is my talk!)

Non-perturbative arguments allow us to check the correctness of the mapping, but only for $Q \neq 0$ and for special observables!

They will never provide a general demonstration that such a change of variables actually works!

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However, if a true equivalence holds, it must be manifest order by order in perturbation theory!

But perturbation theory lives on trivial bundles ($Q = 0$), we have no zero modes that contribute to β!

Where can we look for missing contributions?

Effective action

We must go back to the very start.
The generating functional $Z[J]$ of pure YM is

$$Z\left[J\right] = \int \delta A \, \exp\left[-\frac{1}{2g^2} S_{\text{YM}}\left[A\right] + JA\right]$$

The effective action $\Gamma[\tilde{A}]$ is

$$\exp\left(-\Gamma[\tilde{A}]\right) = Z\left[J\right] \exp(-J\tilde{A}) = \int \delta A \, \exp\left[-\frac{1}{2g^2} S_{\text{YM}}\left[A\right] + JA - J\tilde{A}\right]$$

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We expand the exponent

$$-\frac{1}{2g^2} S_{\rm YM}[\tilde{A}] - \frac{1}{2g^2} \frac{\delta S_{\rm YM}}{\delta A}\bigg|_{\tilde{A}} \delta A - \frac{1}{4g^2} \frac{\delta^2 S_{\rm YM}}{\delta A^2}\bigg|_{\tilde{A}} \delta A^2 + J\delta A$$

cancelled by EOM

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$$-\frac{1}{2g^2} S_{\mathrm{YM}}[\tilde{A}] - \frac{1}{4g^2} \left.\frac{\delta^2 S_{\mathrm{YM}}}{\delta A^2}\right|_{\tilde{A}} \delta A^2$$

The external current does not contribute

Effective action

$$-\frac{1}{2g^2}S_{\text{YM}}[\tilde{A}] - \frac{1}{4g^2}\left.\frac{\delta^2 S_{\text{YM}}}{\delta A^2}\right|_{\tilde{A}}\delta A^2$$

The external current does not contribute

This is true because the external source J couples linearly to the field A

Performing a nonlinear map, this is not true anymore!

Spinorial notation

We introduce a spinorial $SU(2) \otimes SU(2)$ notation

$$A_{\alpha\dot{\alpha}} = A_m \left(\sigma^m\right)_{\alpha\dot{\alpha}} \qquad \bar{A}^{\dot{\alpha}\alpha} = A_m \left(\bar{\sigma}^m\right)^{\dot{\alpha}\alpha}$$
$$D_{\alpha\dot{\alpha}} = D_m \left(\sigma^m\right)_{\alpha\dot{\alpha}} \qquad \bar{D}^{\dot{\alpha}\alpha} = D_m \left(\bar{\sigma}^m\right)^{\dot{\alpha}\alpha}$$

with

$$\left(\sigma^m\right)_{\alpha\dot{\alpha}} = \left(\mathbb{1}, i\vec{\tau}\right)_{\alpha\dot{\alpha}}$$
$$\left(\bar{\sigma}^m\right)^{\dot{\alpha}\alpha} = \left(\mathbb{1}, -i\vec{\tau}\right)^{\dot{\alpha}\alpha}$$

Spinorial notation

We introduce a spinorial $SU(2) \otimes SU(2)$ notation

$$(\mu)^{\dot\alpha}{}_{\dot\beta} = \frac{1}{2} (\mu)_{mn} (\bar\sigma^{mn})^{\dot\alpha}{}_{\dot\beta} \quad \mu \text{ anti-hermitian traceless}$$

$$(\nu)^{\dot\alpha}{}_{\dot\beta} = (\mu)^{\dot\alpha}{}_{\dot\beta} + c\, \delta^{\dot\alpha}_{\dot\beta} \qquad c = D_m(\tilde A)A_m \text{ scalar auxiliary field}$$

$$(\sigma^{mn})_\alpha{}^\beta = \frac{1}{4} \left[(\sigma^m)_{\alpha\dot\alpha} (\bar\sigma^n)^{\dot\alpha\beta} - (\sigma^n)_{\alpha\dot\alpha} (\bar\sigma^m)^{\dot\alpha\beta} \right]$$

$$(\bar\sigma^{mn})^{\dot\alpha}{}_{\dot\beta} = \frac{1}{4} \left[(\bar\sigma^m)^{\dot\alpha\alpha} (\sigma^n)_{\alpha\dot\beta} - (\bar\sigma^n)^{\dot\alpha\alpha} (\sigma^m)_{\alpha\dot\beta} \right]$$

σ^{mn} and $\bar\sigma^{mn}$ project onto *SD* and *ASD* states

The explicit calculation in covariant α-gauges

The generating functional in α-gauge in terms of A_m is

$$Z[J] = \int \delta A \, \delta c \, \exp\left[-\frac{1}{4g^2} \int d^4x \, \mathrm{tr}\left(F^-_{mn}\right)^2 - \frac{1}{\alpha g^2} \int d^4x \, \mathrm{tr}(c^2) + \int d^4x \, 2\,\mathrm{tr}\, J_m A_m \right] \Delta_{\mathrm{FP}} \, \delta\Big(D_m(\tilde{A}) A_m - c \Big)$$

$\tilde{A}$ is a classical field satisfying the EOM

$$J = \frac{1}{2g^2} \left. \frac{\delta S_{\mathrm{YM}}}{\delta A} \right|_{\tilde{A}}$$

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We insert a resolution of the identity
The resolution and the gauge-fixing condition realize
the change of variables $A_m \to (\mu, c) = \nu$

The explicit calculation in covariant α-gauges

The map in spinorial indices reads

$$\nu^a = \bar{\partial} A^a - \frac{1}{2} f^{abc} \bar{A}^b A^c - f^{abc} \tilde{A}^b_m \delta A^c_m \mathbb{1}$$

and the generating functional

$$Z[J] = \int \delta\nu \, \exp\left[-\frac{1}{4g^2} \int d^4x \, (\nu^a)^{\dot\alpha}{}_{\dot\beta} \overbrace{\left[\delta^{\dot\gamma}_{\dot\alpha} \delta^{\dot\beta}_{\dot\delta} - \tfrac{1}{2}\left(1 - \tfrac{1}{\alpha}\right) \delta^{\dot\beta}_{\dot\alpha} \delta^{\dot\delta}_{\dot\gamma} \right]}^{\mathbb{1}_{\frac{1}{\alpha}}} (\nu^a)_{\dot\gamma}{}^{\dot\delta} \right.$$
$$\left. + \int d^4x \, \underbrace{\left(\bar{J}^a\right)^{\dot\alpha\alpha} [A(\nu)]^a_{\alpha\dot\alpha}}_{\text{This contains } O\left(\delta\nu^2\right)} \right] \mathrm{Det} \frac{\delta A}{\delta\nu} \, \Delta_{\mathrm{FP}}$$

The explicit calculation in covariant α-gauges

If we expand $A = \tilde{A} + \delta A$, we get

$$\delta\nu^a = \bar{D}^{ac}\delta A^c - \frac{1}{2} f^{abc}\overline{\delta A^b}\delta A^c$$

that means

$$\left.\frac{\delta\nu}{\delta A}\right|_{\tilde{\nu}} = \bar{D}$$

We can perturbatively invert $\left(G = (-\Delta)^{-1}\,\mathbb{1}\right)$

$$\delta A^{a(1)} = -G\, D^{ac}\,\delta\nu^c$$

$$\delta A^{a(2)} = -\frac{1}{2} f^{abc}\, G\,\partial\left(G\delta\nu^b\,\overleftarrow{\bar{\partial}}\,G\partial\delta\nu^c\right)$$

The explicit calculation in covariant α-gauges

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The EOM is

$$\frac{1}{2g^2}\,\tilde{\nu}^{\dot{\alpha}}{}_{\dot{\beta}}\,\mathbb{1}_{\frac{1}{\alpha}} = \bar{J}^{\dot{\gamma}\delta}\left.\frac{\delta A_{\delta\dot{\gamma}}}{\delta\nu_{\dot{\alpha}}{}^{\dot{\beta}}}\right|_{\tilde{\nu}}$$

The explicit calculation in covariant α-gauges

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$$\frac{1}{2g^2}\,\tilde{\mu}^{\dot{\alpha}}{}_{\dot{\beta}} = \bar{J}^{\dot{\gamma}\delta}\left.\frac{\delta A_{\delta\dot{\gamma}}}{\delta\nu_{\dot{\alpha}}{}^{\dot{\beta}}}\right|_{\tilde{\mu}}$$

We choose the background field to be transverse, i.e. $\tilde{c} = 0$, $\tilde{\nu} = \tilde{\mu}$

The explicit calculation in covariant α-gauges

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$$\left.\frac{\delta\nu}{\delta A}\right|_{\tilde{\nu}} = \bar{D} \qquad \delta A^{a(2)} = -\frac{1}{2} f^{abc}\, G\,\partial\left(G\delta\nu^b \overleftarrow{\bar{\partial}}\, G\partial\delta\nu^c\right)$$

We invert the EOM

$$\bar{J}^{\dot{\gamma}\delta} = -\frac{1}{2g^2}\,\tilde{\mu}^{\dot{\gamma}}{}_{\dot{\beta}}\,\overleftarrow{\bar{D}}^{\dot{\beta}\delta}$$

The explicit calculation in covariant α-gauges

$$\left.\frac{\delta\nu}{\delta A}\right|_{\tilde\nu} = \bar D \qquad \delta A^{a(2)} = -\frac{1}{2} f^{abc}\, G\,\partial\left(G\delta\nu^b \overleftarrow{\bar\partial}\, G\partial\delta\nu^c\right)$$

We invert the EOM

$$\bar J^{\dot\gamma\delta} = -\frac{1}{2g^2}\,\tilde\mu^{\dot\gamma}{}_{\dot\beta}\,\overleftarrow{\bar D}{}^{\dot\beta\delta}$$

Hence, the quadratic part of JA is

$$\begin{aligned}\left(\operatorname{tr}\bar J^a\delta A^a\right)^{(2)} &= \frac{1}{4g^2} f^{abc}\operatorname{tr}\tilde\mu^a\,\overleftarrow{\bar\partial}\,G\partial\left(G\delta\nu^b\overleftarrow{\bar\partial}\,G\partial\delta\nu^c\right)\\ &= -\frac{1}{4g^2} f^{abc}\operatorname{tr}\tilde\mu^a\delta\nu^b\left(\bar\partial G\right)G\partial\delta\nu^c \equiv \frac{1}{4g^2}\delta\nu^b O^{bc}\delta\nu^c\end{aligned}$$

The explicit calculation in covariant α-gauges

We now integrate over $\delta\nu$ to get

$$\int \delta\nu \exp\left[-\frac{1}{2g^2}S_{\text{YM}} + JA\right] = \text{Det}^{-\frac{1}{2}}\left[-\,\mathbb{1}_{\frac{1}{\alpha}}\,\delta^{bc} + O^{bc}\right]$$

We write the Jacobian as

$$\text{Det}\,\frac{\delta A}{\delta\nu} = \text{Det}^{-1}\left(\bar{D}\right) = \text{Det}^{-\frac{1}{2}}\left(D\bar{D}\right)$$

Therefore, the product of the determinants reads

$$\left[\text{Det}\,(D)\,\text{Det}\left(-\,\mathbb{1}_{\frac{1}{\alpha}} + O\right)\text{Det}\left(\bar{D}\right)\right]^{-\frac{1}{2}} = \text{Det}^{-\frac{1}{2}}\left[-D\,\mathbb{1}_{\frac{1}{\alpha}}\,\bar{D} + DO\bar{D}\right]$$

and working out in detail the spinor notation

$$\text{Det}^{-\frac{1}{2}}\left[-\left(D\bar{D}\right)^{\gamma}_{\rho}\delta^{\dot{\alpha}}_{\dot{\beta}} + \tfrac{1}{2}\left(1-\tfrac{1}{\alpha}\right)D_{\gamma\dot{\beta}}\bar{D}^{\dot{\alpha}\rho} + D^{z}_{\gamma\dot{\gamma}}\left[O^{bc}_{zt}\right]^{\dot{\gamma}\dot{\alpha}}_{\dot{\beta}\dot{\rho}}\bar{D}^{\dot{\rho}\rho}_{t}\right]$$

The explicit calculation in covariant α-gauges

and working out in detail the spinor notation

$$\mathrm{Det}^{-\frac{1}{2}}\left[-\left(D\bar{D}\right)_{\rho}^{\gamma}\delta_{\dot\beta}^{\dot\alpha}+\tfrac{1}{2}\left(1-\tfrac{1}{\alpha}\right)D_{\gamma\dot\beta}\bar{D}^{\dot\alpha\rho}+D^{z}_{\gamma\dot\gamma}\left[O^{bc}_{zt}\right]^{\dot\gamma\dot\alpha}_{\dot\beta\dot\rho}\bar{D}^{\dot\rho\rho}_{t}\right]$$

but $D\bar{D}=\Delta\mathbb{1}+i\,\mathrm{ad}\,F^{+}$ and $DO\bar{D}=\mu$, so

$$\mathrm{Det}^{-\frac{1}{2}}\left[-\Delta\delta^{\gamma}_{\rho}\delta^{\dot\alpha}_{\dot\beta}+f^{abc}(F^{+a})^{\gamma}_{\rho}\delta^{\dot\alpha}_{\dot\beta}+\tfrac{1}{2}\left(1-\tfrac{1}{\alpha}\right)\bar{D}^{\dot\alpha\rho}D_{\gamma\dot\beta}+f^{abc}\left(\mu^{a}\right)^{\dot\alpha}_{\dot\beta}\delta^{\gamma}_{\rho}\right]$$

In vector notation

$$\mathrm{Det}^{-\frac{1}{2}}\left[-\Delta\delta_{mn}+\left(1-\frac{1}{\alpha}\right)D_mD_n+\underbrace{f^{abc}F^{+a}_{mn}+f^{abc}\mu^{a}_{mn}}_{F^{+}+\mu=2F}\right]$$

The explicit calculation in covariant α-gauges

and working out in detail the spinor notation

$$\mathrm{Det}^{-\frac{1}{2}}\left[-\left(D\bar{D}\right)_{\rho}^{\gamma}\delta_{\dot{\beta}}^{\dot{\alpha}}+\frac{1}{2}\left(1-\frac{1}{\alpha}\right)D_{\gamma\dot{\beta}}\bar{D}^{\dot{\alpha}\rho}+D^{z}_{\gamma\dot{\gamma}}\left[O^{bc}_{zt}\right]^{\dot{\gamma}\dot{\alpha}}_{\dot{\beta}\dot{\rho}}\bar{D}^{\dot{\rho}\rho}_{t}\right]$$

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$$\mathrm{Det}^{-\frac{1}{2}}\left[-\Delta\delta_{\rho}^{\gamma}\delta_{\dot{\beta}}^{\dot{\alpha}}+f^{abc}(F^{+a})_{\rho}^{\gamma}\delta_{\dot{\beta}}^{\dot{\alpha}}+\frac{1}{2}\left(1-\frac{1}{\alpha}\right)\bar{D}^{\dot{\alpha}\rho}D_{\gamma\dot{\beta}}+f^{abc}\left(\mu^{a}\right)_{\dot{\beta}}^{\dot{\alpha}}\delta_{\rho}^{\gamma}\right]$$

In vector notation

$$\mathrm{Det}^{-\frac{1}{2}}\left[-\Delta\delta_{mn}+\left(1-\frac{1}{\alpha}\right)D_{m}D_{n}+2f^{abc}F^{a}_{mn}\right]$$

This is the same determinant obtained integrating over the gauge connection δA in α-gauge

A geometric demonstration

We show a geometric demonstration valid in any gauge

We introduce $1, 2$-forms

$$A = A_m \, dx_m \qquad J = J_m \, dx_m$$

$$F = F_{mn} \, dx_m \wedge dx_n \qquad \mu = \mu_{mn} \, dx_m \wedge dx_n$$

$$F = dA + iA \wedge A$$

The generating functional reads

$$Z[J] = \int \delta\mu \, \exp\left[\frac{1}{8g^2} \int \mu \wedge \mu + \int *J \wedge A\right] \delta\left(\mathcal{G}\left(A\right)\right) \, \text{Det} \, \frac{\delta A}{\delta \mu} \, \Delta_{\text{FP}}$$

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$\delta(\mathcal{G}(A))$ restricts the determinants to a sub-manifold

A geometric demonstration

The map reads

$$\mu = 2P^- F = 2P^- \left(F(\tilde{A}) + d_{\tilde{A}} \wedge \delta A + i\delta A \wedge \delta A \right)$$
$$\tilde{\mu} = 2P^- F(\tilde{A})$$
$$\delta\mu = 2P^- \left(d_{\tilde{A}} \wedge \delta A + i\delta A \wedge \delta A \right)$$

We invert perturbatively

$$\delta A^{(1)} = \frac{1}{2} \left(P^- d_{\tilde{A}} \wedge \right)^{-1} \delta\mu$$
$$\delta A^{(2)} = -\frac{i}{4} \left(P^- d_{\tilde{A}} \wedge \right)^{-1} \left[\left(P^- d_{\tilde{A}} \wedge \right)^{-1} \delta\mu \wedge \left(P^- d_{\tilde{A}} \wedge \right)^{-1} \delta\mu \right]$$

The EOM reads

$$\tilde{\mu} = -4g^2 * J \wedge \left. \frac{\delta A}{\delta\mu} \right|_{\tilde{\mu}}$$

A geometric demonstration

The map reads

$$\mu = 2P^- F = 2P^- \left(F(\tilde{A}) + d_{\tilde{A}} \wedge \delta A + i\delta A \wedge \delta A\right)$$

$$\tilde{\mu} = 2P^- F(\tilde{A})$$

$$\delta\mu = 2P^- \left(d_{\tilde{A}} \wedge \delta A + i\delta A \wedge \delta A\right)$$

We invert perturbatively

$$\delta A^{(1)} = \frac{1}{2}\left(P^- d_{\tilde{A}} \wedge\right)^{-1} \delta\mu$$

$$\delta A^{(2)} = -\frac{i}{4}\left(P^- d_{\tilde{A}} \wedge\right)^{-1} \left[\left(P^- d_{\tilde{A}} \wedge\right)^{-1} \delta\mu \wedge \left(P^- d_{\tilde{A}} \wedge\right)^{-1} \delta\mu\right]$$

The EOM reads

$$J = \frac{1}{2g^2} * \left(P^- d_{\tilde{A}} \wedge \tilde{\mu}\right)$$

A geometric demonstration

$\delta A^{(2)}$ and the saddle point equation determine the quadratic form that contributes to the effective action

$$\exp\left[\frac{1}{8g^2}\int \delta\mu \wedge \delta\mu + \frac{1}{2}\int *J \wedge \frac{\delta^2 A}{\delta\mu^2}\delta\mu^2\right]$$

The red term reads

$$= -\frac{i}{8}\left(P^- d_A \wedge \tilde{\mu}\right)\wedge\left(P^- d_A\wedge\right)^{-1}\left[\left(P^- d_A\wedge\right)^{-1}\delta\mu \wedge \left(P^- d_A\wedge\right)^{-1}\delta\mu\right]$$

A geometric demonstration

$\delta A^{(2)}$ and the saddle point equation determine the quadratic form that contributes to the effective action

$$\exp\left[\frac{1}{8g^2}\int \delta\mu\wedge\delta\mu + \frac{1}{2}\int *J\wedge\frac{\delta^2 A}{\delta\mu^2}\delta\mu^2\right]$$

The red term reads

$$= \frac{i}{8}\tilde{\mu}\wedge\left(P^- d_A\wedge\right)^{-1}\delta\mu\wedge\left(P^- d_A\wedge\right)^{-1}\delta\mu$$

A geometric demonstration

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The red term reads

$$= \frac{i}{8} 2P^- F \wedge \left(P^- d_A\wedge\right)^{-1}\delta\mu\wedge\left(P^- d_A\wedge\right)^{-1}\delta\mu$$

A geometric demonstration

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The red term reads

$$= \frac{i}{8} 2P^- F \wedge \left(P^- d_A \wedge\right)^{-1} \delta\mu \wedge \left(P^- d_A \wedge\right)^{-1} \delta\mu$$

Hence the quadratic form is

$$\frac{1}{8g^2}\left[\int \delta\mu \wedge \delta\mu + 2i \int P^- F \wedge \left(P^- d_A \wedge\right)^{-1} \delta\mu \wedge \left(P^- d_A \wedge\right)^{-1} \delta\mu\right]$$

A geometric demonstration

Integrating over $\delta\mu$,

$$\mathrm{Det}^{-1/2}\left[1+2i\left(P^{-}d_A\wedge\right)^{-1}P^{-}F\wedge\left(P^{-}d_A\wedge\right)^{-1}\right]_{\mathcal{G}=0}$$

The Jacobian reads

$$\mathrm{Det}\,\frac{\delta A}{\delta\mu}=\mathrm{Det}\left(P^{-}d_A\wedge\right)^{-1}=\mathrm{Det}^{-1/2}\left(P^{-}d_A\wedge\right)\mathrm{Det}^{-1/2}\left(P^{-}d_A\wedge\right)$$

All together

$$\mathrm{Det}^{-1/2}\left[P^{-}d_A\wedge P^{-}d_A\wedge+2i\,P^{-}F\wedge\right]_{\mathcal{G}=0}$$

Just like the original determinant in *YM* effective action!

The evaluation of the β function in Feynman gauge

As a check, we evaluate the β function in Feynman gauge, where we can extract the counterterms from each determinant separately

$$Z = e^{-\frac{S_{\mathrm{YM}}}{2g^2}} \int \delta\nu \exp\left[-\frac{1}{2g^2} S_{\mathrm{YM}} + JA\right] \mathrm{Det}\, \frac{\delta A}{\delta \nu}\, \Delta_{\mathrm{FP}}$$

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We evaluate the Jacobian

$$\text{Det}\,\frac{\delta A}{\delta \nu} = \text{Det}^{-1}\left(\bar{D}\right) = \text{Det}^{-\frac{1}{2}}\left(D\bar{D}\right)$$

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We evaluate the Jacobian

$$\text{Det} \, \frac{\delta A}{\delta \nu} = \text{Det}^{-\frac{1}{2}} \left(-\Delta \delta_{mn} + i \, \text{ad} \, F^{+}_{mn} \right)$$

The evaluation of the β function in Feynman gauge

As a check, we evaluate the β function in Feynman gauge, where we can extract the counterterms from each determinant separately

$$Z = e^{-\frac{S_{\rm YM}}{2g^2}}\,\text{Det}^{-\frac{1}{2}}\left[-\mathbb{1}\delta^{bc} + O^{bc}\right]\,\text{Det}\,\frac{\delta A}{\delta \nu}\,\Delta_{\rm FP}$$

We evaluate the Jacobian

$$\text{Det}\,\frac{\delta A}{\delta \nu} = \text{Det}^{-\frac{1}{2}}\left(-\Delta\delta_{mn}\right)\text{Det}^{-\frac{1}{2}}\left(1 + (-\Delta)^{-1}\, i\,\text{ad}\,F^{+}_{mn}\right)$$

We can evaluate the orbital term

$$\text{Det}^{-\frac{1}{2}}\left(-\Delta\delta_{mn}\right)\underbrace{\text{Det}\left(-\Delta\right)}_{\Delta_{\rm FP}} = \exp\left[-\frac{\frac{N}{3}}{(4\pi)^2}\,\log\frac{\Lambda}{\mu}\,S_{\rm YM}\right]$$

The evaluation of the β function in Feynman gauge

As a check, we evaluate the β function in Feynman gauge, where we can extract the counterterms from each determinant separately

$$Z = e^{-\frac{S_{\mathrm{YM}}}{2g^2}}\,\mathrm{Det}^{-\frac{1}{2}}\left[-\mathbb{1}\delta^{bc} + O^{bc}\right]\,\mathrm{Det}\,\frac{\delta A}{\delta\nu}\,\Delta_{\mathrm{FP}}$$

We evaluate the Jacobian

$$\mathrm{Det}\,\frac{\delta A}{\delta\nu} = \mathrm{Det}^{-\frac{1}{2}}\left(-\Delta\delta_{mn}\right)\mathrm{Det}^{-\frac{1}{2}}\left(1 + (-\Delta)^{-1}\,i\,\mathrm{ad}\,F^{+}_{mn}\right)$$

As for the spin term ($\frac{1}{2}$ of pure YM spin term)

$$\mathrm{Det}^{-\frac{1}{2}}\left(1 + (-\Delta)^{-1}\,\mathrm{ad}\,F^{+}_{mn}\right) = \exp\left[\frac{2N}{(4\pi)^2}\,\log\frac{\Lambda}{\mu}\,S_{\mathrm{YM}}\right]$$

The evaluation of the β function in Feynman gauge

Finally, the extra determinant

$$\text{Det}^{-\frac{1}{2}}\left(-\mathbb{1}+O\right)=\exp\left[\frac{2N}{(4\pi)^2}\,\log\frac{\Lambda}{\mu}\,S_{\text{YM}}\right]$$

provides the missing term

$$-\frac{S_{\text{YM}}}{2g^2(\mu)}=-\frac{S_{\text{YM}}}{2g^2(\Lambda)}-\frac{\frac{N}{3}-2N-2N}{(4\pi)^2}\,\log\frac{\Lambda}{\mu}\,S_{\text{YM}}$$

The evaluation of the β function in Feynman gauge

Finally, the extra determinant

$$\mathrm{Det}^{-\frac{1}{2}}\left(-\mathbb{1}+O\right)=\exp\left[\frac{2N}{(4\pi)^2}\log\frac{\Lambda}{\mu}\,S_{\mathrm{YM}}\right]$$

provides the missing term

$$\beta=-\frac{11N}{3}\frac{g^3}{(4\pi)^2}$$

Conclusions

- We have established the perturbative equivalence of gauge theories usually formulated in terms of the connection A_m with gauge theories formulated in terms of the *ASD* curvature F^-_{mn}
- We have evaluated the 1-loop effective action of the mapped theory and shown that is identical to the original one
- This argument generalizes order by order in perturbation theory, since the map is perturbatively invertible at any given order
- *ASD* variables are a new language to describe gauge theories whose potentiality remains to be fully explored

Conclusions

- We have established the perturbative equivalence of gauge theories usually formulated in terms of the connection A_m with gauge theories formulated in terms of the *ASD* curvature F^-_{mn}
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- This argument generalizes order by order in perturbation theory, since the map is perturbatively invertible at any given order
- *ASD* variables are a new language to describe gauge theories whose potentiality remains to be fully explored

Thank you

Backup slides

A sketch of β function in pure YM

An example of non-perturbative applications is the evaluation of large-N pure YM β function by homological localization of twistor Wilson loops (Bochicchio, JHEP 0905 (2009) 116)

$$Z = \int \delta\mu \, e^{-\frac{(4\pi)^2 Q}{2g_W^2}} \exp\left(-\frac{1}{8Ng_W^2}\int d^4x \,\mathrm{tr}\,(\mu_{mn})^2\right)$$
$$\mathrm{Det}\,\frac{\delta A}{\delta\mu}\,\Delta_{\mathrm{FP}}\,\Lambda^{n_b[A]}\int_{\mathcal{M}} \mathrm{Pf}\left\langle \frac{\delta A(\mu)}{\delta m}, \frac{\delta A(\mu)}{\delta m}\right\rangle$$

In Feynman gauge,

$$\mathrm{Det}\,\frac{\delta A}{\delta\mu}\,\Delta_{\mathrm{FP}} \Rightarrow \beta_0^{(J,FP)} = \frac{-\frac{5}{3}N}{(4\pi)^2}$$

We can localize the functional integral on singular configurations $F^- = \sum_P \mu_P \delta(x - x_P)$

A sketch of β function in pure YM

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$$\operatorname{Det}\frac{\delta A}{\delta\mu}\,\Delta_{\mathrm{FP}}\,\Lambda^{n_b[A]}\int_{\mathcal{M}} \operatorname{Pf}\left\langle \frac{\delta A(\mu)}{\delta m}, \frac{\delta A(\mu)}{\delta m}\right\rangle$$

We take into account the contribution from zero modes

$$\Lambda^{n_b[A]} \Rightarrow \beta_0^{(0)} = -\frac{2N}{(4\pi)^2}$$

We get the correct Wilsonian β function $\beta_W = -\frac{\frac{11}{3}N}{(4\pi)^2}g^3$

A sketch of β function in pure YM

For the canonical β function, we rescale the fields and get

$$\beta = \frac{-\frac{11N}{3}\frac{g^3}{(4\pi)^2} + \frac{Ng^3}{(4\pi)^2}\frac{\partial \log Z}{\log \Lambda}}{1 - \frac{4N}{(4\pi)^2}g^2}$$
$$\sim -\frac{11N}{3}\frac{g^3}{(4\pi)^2} - \frac{34N^2}{3}\frac{g^5}{(4\pi)^4} + \cdots$$

that reproduces the universal coefficients of perturbative β function